



FHSST Authors

**The Free High School Science Texts:
Textbooks for High School Students
Studying the Sciences
Mathematics
Grades 10 - 12**

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Chapter 28

Exponential Functions and Graphs - Grade 11

28.1 Introduction

In Grade 10, you studied graphs of many different forms. In this chapter, you will learn a little more about the graphs of exponential functions.

28.2 Functions of the Form $y = ab^{(x+p)} + q$

This form of the exponential function is slightly more complex than the form studied in Grade 10.

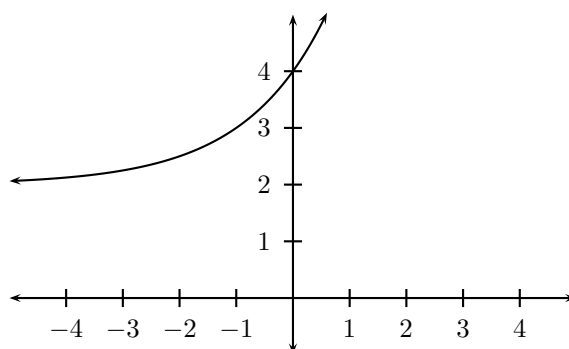


Figure 28.1: General shape and position of the graph of a function of the form $f(x) = ab^{(x+p)} + q$.

Activity :: Investigation : Functions of the Form $y = ab^{(x+p)} + q$

1. On the same set of axes, plot the following graphs:

A $a(x) = -2 \cdot b^{(x+1)} + 1$

B $b(x) = -1 \cdot b^{(x+1)} + 1$

C $c(x) = -0 \cdot b^{(x+1)} + 1$

D $d(x) = -1 \cdot b^{(x+1)} + 1$

E $e(x) = -2 \cdot b^{(x+1)} + 1$

Use your results to deduce the effect of a .

2. On the same set of axes, plot the following graphs:

- A $f(x) = 1 \cdot b^{(x+1)} - 2$
 B $g(x) = 1 \cdot b^{(x+1)} - 1$
 C $h(x) = 1 \cdot b^{(x+1)} 0$
 D $j(x) = 1 \cdot b^{(x+1)} + 1$
 E $k(x) = 1 \cdot b^{(x+1)} + 2$

Use your results to deduce the effect of q .

3. Following the general method of the above activities, choose your own values of a and q to plot 5 different graphs of $y = ab^{(x+p)} + q$ to deduce the effect of p .

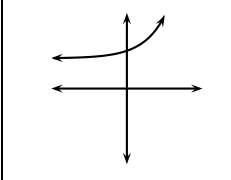
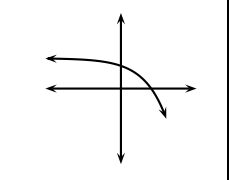
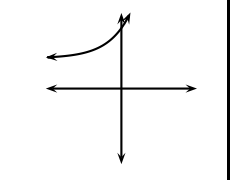
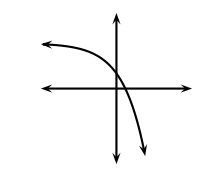
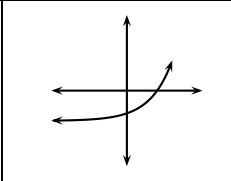
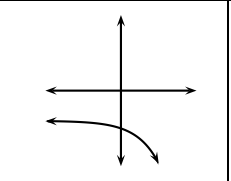
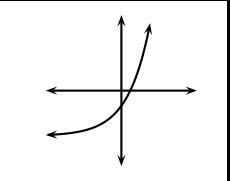
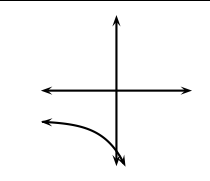
You should have found that the value of a affects whether the graph curves upwards ($a > 0$) or curves downwards ($a < 0$).

You should have also found that the value of p affects the position of the x -intercept.

You should have also found that the value of q affects the position of the y -intercept.

These different properties are summarised in Table 28.1. The axes of symmetry for each graph is shown as a dashed line.

Table 28.1: Table summarising general shapes and positions of functions of the form $y = ab^{(x+p)} + q$.

	$p < 0$		$p > 0$	
	$a > 0$	$a < 0$	$a > 0$	$a < 0$
$q > 0$				
$q < 0$				

28.2.1 Domain and Range

For $y = ab^{(x+p)} + q$, the function is defined for all real values of x . Therefore, the domain is $\{x : x \in \mathbb{R}\}$.

The range of $y = ab^{(x+p)} + q$ is dependent on the sign of a .

If $a > 0$ then:

$$\begin{aligned} b^{(x+p)} &\geq 0 \\ a \cdot b^{(x+p)} &\geq 0 \\ a \cdot b^{(x+p)} + q &\geq q \\ f(x) &\geq q \end{aligned}$$

Therefore, if $a > 0$, then the range is $\{f(x) : f(x) \in [q, \infty)\}$.

If $a < 0$ then:

$$\begin{aligned} b^{(x+p)} &\leq 0 \\ a \cdot b^{(x+p)} &\leq 0 \\ a \cdot b^{(x+p)} + q &\leq q \\ f(x) &\leq q \end{aligned}$$

Therefore, if $a < 0$, then the range is $\{f(x) : f(x) \in (-\infty, q]\}$.

For example, the domain of $g(x) = 3 \cdot 2^{x+1} + 2$ is $\{x : x \in \mathbb{R}\}$. For the range,

$$\begin{aligned} 2^{x+1} &\geq 0 \\ 3 \cdot 2^{x+1} &\geq 0 \\ 3 \cdot 2^{x+1} + 2 &\geq 2 \end{aligned}$$

Therefore the range is $\{g(x) : g(x) \in [2, \infty)\}$.



Exercise: Domain and Range

1. Give the domain of $y = 3^x$.
 2. What is the domain and range of $f(x) = 2^x$?
 3. Determine the domain and range of $y = (1,5)^{x+3}$.
-

28.2.2 Intercepts

For functions of the form, $y = ab^{(x+p)} + q$, the intercepts with the x and y axis is calculated by setting $x = 0$ for the y -intercept and by setting $y = 0$ for the x -intercept.

The y -intercept is calculated as follows:

$$y = ab^{(x+p)} + q \quad (28.1)$$

$$y_{int} = ab^{(0+p)} + q \quad (28.2)$$

$$= ab^p + q \quad (28.3)$$

For example, the y -intercept of $g(x) = 3 \cdot 2^{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^{x+1} + 2 \\ y_{int} &= 3 \cdot 2^{0+1} + 2 \\ &= 3 \cdot 2^1 + 2 \\ &= 3 \cdot 2 + 2 \\ &= 8 \end{aligned}$$

The x -intercepts are calculated by setting $y = 0$ as follows:

$$y = ab^{(x+p)} + q \quad (28.4)$$

$$0 = ab^{(x_{int}+p)} + q \quad (28.5)$$

$$ab^{(x_{int}+p)} = -q \quad (28.6)$$

$$b^{(x_{int}+p)} = -\frac{q}{a} \quad (28.7)$$

Which only has a real solution if either $a < 0$ or $Q < 0$. Otherwise, the graph of the function of form $y = ab^{(x+p)} + q$ does not have any x -intercepts.

For example, the x -intercept of $g(x) = 3 \cdot 2^{x+1} + 2$ is given by setting $x = 0$ to get:

$$\begin{aligned} y &= 3 \cdot 2^{x+1} + 2 \\ 0 &= 3 \cdot 2^{x_{int}+1} + 2 \\ -2 &= 3 \cdot 2^{x_{int}+1} \\ 2^{x_{int}+1} &= \frac{-2}{3} \end{aligned}$$

which has no real solution. Therefore, the graph of $g(x) = 3 \cdot 2^{x+1} + 2$ does not have any x -intercepts.



Exercise: Intercepts

1. Give the y -intercept of the graph of $y = b^x + 2$.
 2. Give the x - and y -intercepts of the graph of $y = \frac{1}{2}(1,5)^{x+3} - 0,75$.
-

28.2.3 Asymptotes

There are two asymptotes for functions of the form $y = ab^{(x+p)} + q$. They are determined by examining the domain and range.

We saw that the function was undefined at $x = -p$ and for $y = q$. Therefore the asymptotes are $x = -p$ and $y = q$.

For example, the domain of $g(x) = 3 \cdot 2^{x+1} + 2$ is $\{x : x \in \mathbb{R}, x \neq -1\}$ because $g(x)$ is undefined at $x = -1$. We also see that $g(x)$ is undefined at $y = 2$. Therefore the range is $\{g(x) : g(x) \in (-\infty, 2) \cup (2, \infty)\}$.

From this we deduce that the asymptotes are at $x = -1$ and $y = 2$.



Exercise: Asymptotes

1. Give the equation of the asymptote of the graph of $y = 3^x - 2$.
 2. What is the equation of the horizontal asymptote of the graph of $y = 3(0,8)^{x-1} - 3$?
-

28.2.4 Sketching Graphs of the Form $f(x) = ab^{(x+p)} + q$

In order to sketch graphs of functions of the form, $f(x) = ab^{(x+p)} + q$, we need to calculate determine four characteristics:

1. domain and range
2. y -intercept

3. x -intercept

For example, sketch the graph of $g(x) = 3 \cdot 2^{x+1} + 2$. Mark the intercepts.

We have determined the domain to be $\{x : x \in \mathbb{R}\}$ and the range to be $\{g(x) : g(x) \in [5, \infty)\}$.

The y -intercept is $y_{int} = 8$ and there are no x -intercepts.

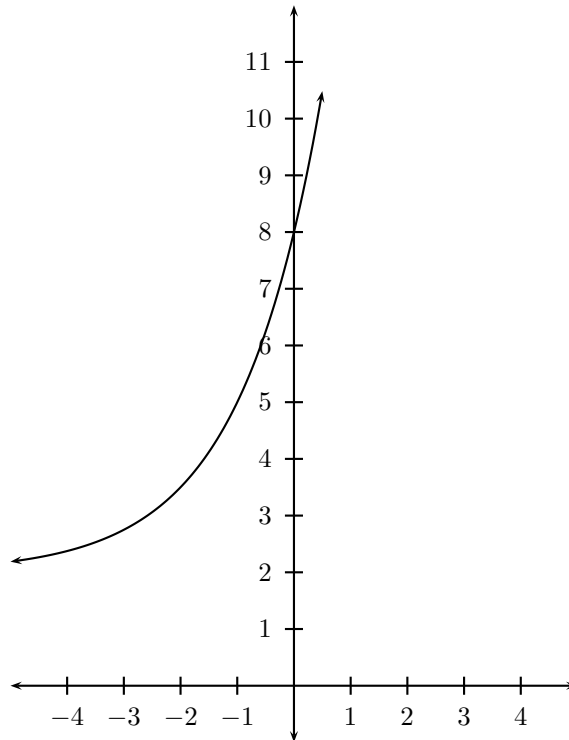


Figure 28.2: Graph of $g(x) = 3 \cdot 2^{x+1} + 2$.



Exercise: Sketching Graphs

1. Draw the graphs of the following on the same set of axes. Label the horizontal asymptotes and y -intercepts clearly.

- A $y = b^x + 2$
- B $y = b^{x+2}$
- C $y = 2b^x$
- D $y = 2b^{x+2} + 2$

- A Draw the graph of $f(x) = 3^x$.
 - B Explain where a solution of $3^x = 5$ can be read off the graph.
-

28.3 End of Chapter Exercises

1. The following table of values has columns giving the y -values for the graph $y = a^x$, $y = a^{x+1}$ and $y = a^x + 1$. Match a graph to a column.

x	A	B	C
-2	7,25	6,25	2,5
-1	3,5	2,5	1
0	2	1	0,4
1	1,4	0,4	0,16
2	1,16	0,16	0,064

2. The graph of $f(x) = 1 + a \cdot 2^x$ (a is a constant) passes through the origin.
- A Determine the value of a .
 - B Determine the value of $f(-15)$ correct to FIVE decimal places.
 - C Determine the value of x , if $P(x; 0,5)$ lies on the graph of f .
 - D If the graph of f is shifted 2 units to the right to give the function h , write down the equation of h .
3. The graph of $f(x) = a \cdot bx$ ($a \neq 0$) has the point $P(2;144)$ on f .
- A If $b = 0,75$, calculate the value of a .
 - B Hence write down the equation of f .
 - C Determine, correct to TWO decimal places, the value of $f(13)$.
 - D Describe the transformation of the curve of f to h if $h(x) = f(-x)$.

Appendix A

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